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Efficient finite-difference scheme for solving some heat transfer problems with convection in multilayer media

H. Kalis

Department of Physics and Mathematics, University of Latvia, 19 Blv. Rainis, Riga, LV-1586, Latvia

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Abstract

An efficient finite-difference method for solving the heat transfer equation with piecewise discontinuous coefficients in a multilayer domain is developed. The method may be considered as a generalization of the finite-volumes method for the layered systems. We apply this method with the aim to reduce the 3D or 2D problem to the corresponding series of 2D or 1D problems. In the case of constant piecewise coefficients, we obtain the exact discrete approximation of the steady-state 1D boundary-value problem. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

For the mathematical modelling of the heat transfer process in multilayer materials special averaging procedures are considered [1,2]. A specific feature of such problems is the necessity to solve the 3D or 2D initial-boundary value problem for a parabolic type partial differential equation with piecewise discontinuous coefficients in the same thin layers. The averaging method of solving these problems by means of quadratic polinomials leads to a situation when boundary conditions contain terms of higher order than the differential equation. This causes additional difficulties for the applications of general difference methods. That is why it is important to work out special methods of solution.

2. Formulation of the problem

We shall consider the process of heat transfer in a 3D cylindrical domain

$$D = \{(x, y, z):(x, y) \in \Omega, H_0 \le z \le H_N\},$$

where $\Omega = \{(x, y): -l_x \le x \le l_x, \ -l_y \le y \le l_y\}$ is a rectangle in the horizontal x-, y-directions with length of edges $2l_x$, $2l_y$, $H_N - H_0$ is the height of the domain in the vertical z-direction. Domain D consists of an N-layer medium

$$D_k = \{(x, y, z):(x, y) \in \Omega, H_{k-1} < z < H_k\} \quad k = 1, \ldots, N, \tag{1}$$

with horizontal interfaces

$$S_k = \{(x, y, H_k):(x, y) \in \Omega\} \quad k = 1, \ldots, N-1, \tag{2}$$

where $H_k - H_{k-1}$ is the height of layer D_k.

We will find the distribution of temperature field $u_k = u_k(x, y, z, t)$ in every layer D_k at a point $(x, y, z) \in D_k$ and time $t > 0$ by solving the partial differential equation of the following form:

$$\rho_k c_k(\partial u_k/\partial t + \mathbf{w}_k \operatorname{grad} u_k) = \operatorname{div}(\kappa_k \operatorname{grad} u_k) + q_k, \quad k = 1, \ldots, N,$$

E-mail address: kalis@fmf.lu.lv (H. Kalis).

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where ρ_k, c_k, κ_k are the coefficients of density, heat capacity and heat conductivity, respectively. $q_k = q_k(x, y, z, t)$ is the heat source function and $\mathbf{w}_k$ is the given velocity vector with components (w_k^x, w_k^y, w_k^z).

We assume that these components, as well as other physical parameters in the equation are depending only on x, y, t and are piecewise continuous functions of the vertical coordinate z with discontinuity points on surface S_k.

We can consider a differential equation of the form

$$\partial(\lambda_k \partial u_k/\partial z)/\partial z - e_k \partial u_k/\partial z + L_k(u_k) = -F_k(x, y, z, t), \tag{3}$$

where $F_k = q_k/(\rho_k c_k)$ are continuously-differentiable functions of external sources in every layer D_k, $e_k = w_k^z$.

The differential operator L_k we can write in the form

$$L_k(u_k) = \partial(\lambda_k \partial u_k/\partial x)/\partial x + \partial(\lambda_k \partial u_k/\partial y)/\partial y - d_k \partial u_k/\partial t - a_k \partial u_k/\partial x - b_k \partial u_k/\partial y \tag{4}$$

$$k = 1, \ldots, N,$$

where $d_k = 1$, $a_k = w_k^x$, $b_k = w_k^y$, $\lambda_k = \kappa/(\rho_k c_k)$.

Eqs. (3) and (4) should be written for every layer D_k having different properties of material. Temperature u_k and heat flux $\lambda_k \partial u_k/\partial z$ must be continuous on the interior boundary S_k. Therefore, we have the following continuity conditions on this boundary

$$u_k = u_{k+1},$$
$$\lambda_k \partial u_k/\partial z = \lambda_{k+1} \partial u_{k+1}/\partial z, \quad k = 1, \ldots, N-1. \tag{5}$$

We assume that the whole N-layer system is bounded from above and below by the plane surfaces S_0, S_N(2). The boundary conditions on these surfaces may be written as

$$\nu_0 \lambda_1 \partial u_1/\partial z - \alpha_0 u_1 = -\alpha_0 \Phi_0(x, y, t), \tag{6}$$

$$\nu_1 \lambda_N \partial u_N/\partial z + \alpha_N u_N = \alpha_N \Phi_1(x, y, t), \tag{7}$$

where $(x, y) \in \Omega$, $t \geq 0$, $\nu_0 = 0$ or $\nu_1 = 0$ for the corresponding Dirichlet boundary conditons: $u_1 = \Phi_0$ or $u_N = \Phi_1$; $\nu_0 = 1$ or $\nu_1 = 1$ for the corresponding Neumann ($\alpha_0 = 0$ or $\alpha_N = 0$) or general form of boundary conditions; $\alpha_0 \geq 0, \alpha_N \geq 0$ are the coefficients of heat transfer, Φ_0, Φ_1 are the functions of external temperature.

Eqs. (3) and (4) with conditions (5)–(7) along the z-coordinate have been solved in the domain D with different boundary conditions in the x-, y-directions at $x = \pm l_x$, $y = \pm l_y$ and with an initial condition at $t = 0$ in the case of time depending problem. The form of this condition are not essential for obtaining a numerical algorithm.

3. The finite-difference approximations

The approximation of differential problem is based on the conservation law approach. Therefore, it develops the monotone difference scheme using a heat conservation law. This method is based on the application of the method of finite volumes [3]. We consider a nonuniform grid in the z-direction placed in the interval (H_0, H_N) with blocks centered at the grid points z_j, $j = 1, \ldots, M$, $M \geq N$ $(z_0 = H_0, z_M = H_N)$. We shall refer to the endpoints of the interval about the point z_j as $z_{j\pm 0.5}$. This interval $(z_{j-0.5}, z_{j+0.5})$ is referred to a control volume associated with the grid point z_j (the jth cell). The grid contains z-coordinates H_k of surfaces S_k, $k = 0, \ldots, N$ and, in addition, some grid points in layers D_k, $k = 1, \ldots, N$ when this is necessary for demonstrating the behaviour of the discrete solution in these layers.

To derive a difference equation associated with grid point z_j, we integrate differential equation (3) to the jth cell. For this purpose we apply the self-adjoint form of Eq. (3) to the intervals (z_{j-1}, z_j), (z_j, z_{j+1}).

We have

$$\partial\left(\lambda_j^* \partial u_j/\partial z\right)/\partial z = G_j, \quad z \in (z_{j-1}, z_j),$$

$$\partial\left(\lambda_{j+1}^* \partial u_{j+1}/\partial z\right)/\partial z = G_{j+1}, \quad z \in (z_j, z_{j+1}), \tag{8}$$

where $\lambda_j^* = J_j \lambda_j$, $G_j = -J_j(F_j + L_j(u_j))$, $J_j = \exp(-\beta_j(z - z_j)/h_j)$, $J_{j+1} = \exp(-\beta_{j+1}(z - z_j)/h_{j+1})$, $\beta_j = (\lambda_j)^{-1} e_j h_j$, $h_j = z_j - z_{j-1}$. We define the heat flux $\mathbf{W} = \lambda^* \partial u/\partial z$, $\mathbf{W}_{j\pm 0.5}$ and the integrals

$$B_j = \int_{z_{j-1}}^{z_j} \left(\lambda_j^*\right)^{-1} \mathrm{d}z \int_{z_{j-0.5}}^{z} G_j \, \mathrm{d}\xi,$$

where,

$$\mathbf{W}_{j\pm 0.5} = \mathbf{W}|_{z = z_{j\pm 0.5}}, \quad z_{j\pm 0.5} = (z_j + z_{j\pm 1})/2.$$

We shall consider, from central grid point z_j, the following four cases for applying the finite volumes method: $z_j \in S_k, k = 1, \ldots, N-1$; $z_j \in D_k, k = 1, \ldots, N$; $z_j \in S_0$ if $\nu_0 = 1$ and $z_j \in S_N$ if $\nu_1 = 1$.

3.1 Let $z_j = H_k$ be the central grid point and z_{j-1}, z_{j+1} the other grid points. We integrate Eq. (8) from $z_{j-0.5}$ to $z_{j+0.5}$ and get

$$\mathbf{W}_{j+0.5} - \mathbf{W}_{j-0.5} = \int_{z_{j-0.5}}^{z_j} G_j \, \mathrm{d}z + \int_{z_j}^{z_{j+0.5}} G_{j+1} \, \mathrm{d}z, \tag{9}$$

where $G_j = G_k$, $G_{j+1} = G_{k+1}$. This is the integral form of the conservation law for the interval $(z_{j-0.5}, z_{j+0.5})$. In the classical formulation for the finite volumes method [3] it is assumed that the flux terms $\mathbf{W}_{j\pm0.5}$ in Eq. (9) are approximate with the difference expressions. Then the corresponding difference scheme is not exact for given functions G_j. Here, we have the possibility to make the exact difference scheme.

For this purpose we integrate Eq. (8) from $z_{j-0.5}$ to $z \in (z_{j-1}, z_j)$ and get,

$$\mathbf{W} - \mathbf{W}_{j-0.5} = \int_{z_{j-0.5}}^{z} G_j \, \mathrm{d}\xi.$$

After dividing this expression by λ_k^* and integrating from z_{j-1} to z_j we obtain:

$$u_k(z_j) - u_k(z_{j-1}) = \left(A_j^-\right)^{-1}\mathbf{W}_{j-0.5} + B_j,$$

where $(A_j^-)^{-1} = \int_{z_{j-1}}^{z_j} (\lambda_j^*)^{-1}\,\mathrm{d}z = e_j^{-1}(1-\exp(-\beta_j))$, $e_j = e_k$, $\lambda_j = \lambda_k$ and $u_k(z_j), u_k(z_{j-1})$ represents the values of function u_k at z_j, z_{j-1}.

Hence

$$\mathbf{W}_{j-0.5} = A_j^-(u_k(z_j) - u_k(z_{j-1})) - A_j^- B_j. \tag{10}$$

Similarly, determining the flux term $\mathbf{W}_{j+0.5}$ by integrating Eq. (8) in the intervals $(z_{j+0.5}, z)$, $z \in (z_j, z_{j+1})$ and (z_j, z_{j+1}) one obtains

$$\mathbf{W}_{j+0.5} = A_{j+1}^+(u_{k+1}(z_{j+1}) - u_{k+1}(z_j)) - A_{j+1}^+ B_{j+1}, \tag{11}$$

where $(A_{j+1}^+)^{-1} = \int_{z_j}^{z_{j+1}} (\lambda_{j+1}^*)^{-1}\,\mathrm{d}z = e_{j+1}^{-1}(-1+\exp(\beta_{j+1}))$, $e_{j+1} = e_{k+1}$, $\lambda_{j+1} = \lambda_{k+1}$ and $u_{k+1}(z_j)$, $u_{k+1}(z_{j+1})$ are the values of function u_{k+1} at z_j, z_{j+1}.

To derive a three-point difference equation associated with the central grid point $z_j = H_k$ we have to apply Eq. (9) in the form

$$A_{j+1}^+(u_{k+1}(z_{j+1}) - u_{k+1}(z_j)) - A_j^-(u_k(z_j) - u_k(z_{j-1})) = R_j, \tag{12}$$

where,

$$R_j = A_{j+1}^+ B_{j+1} - A_j^- B_j + \int_{z_{j-0.5}}^{z_j} G_j \, \mathrm{d}z + \int_{z_j}^{z_{j+0.5}} G_{j+1} \, \mathrm{d}z.$$

The integrals B_j, B_{j+1} can be modified by the partial integral formula and the right side of Eq. (12) can be rewritten as

$$R_j = \int_{z_{j-1}}^{z_j} \left(1 - A_j^- \int_z^{z_j} \left(\lambda_j^*\right)^{-1} \mathrm{d}\xi\right) G_j \, \mathrm{d}z + \int_{z_j}^{z_{j+1}} \left(1 - A_{j+1}^+ \int_{z_j}^{z} \left(\lambda_{j+1}^*\right)^{-1} \mathrm{d}\xi\right) G_{j+1} \, \mathrm{d}z. \tag{13}$$

If $g(s) = s(\exp(s)-1)^{-1}$ is a real positive function with properties $g(\infty) = 0$, $g(-\infty) = \infty$, $g(s) = 1 - s/2 + O(s^2)$, then

$$A_j^- = \lambda_j h_j^{-1} g(-\beta_j), \qquad A_{j+1}^+ = \lambda_{j+1} h_{j+1}^{-1} g(\beta_{j+1})$$

and difference equation (12) has the form

$$\lambda_{j+1} h_{j+1}^{-1} g(\beta_{j+1})(u_{k+1}(z_{j+1}) - u_{k+1}(z_j)) - \lambda_j h_j^{-1} g(-\beta_j)(u_k(z_j) - u_k(z_{j-1})) = R_j, \tag{14}$$

where,

$$\beta_j = \lambda_j^{-1} e_j h_j, \qquad \beta_{j+1} = \lambda_{j+1}^{-1} e_{j+1} h_{j+1}.$$

3.2 If $z_j \in D_k, h_j = h_{j+1}$, then $\beta_j = \beta_{j+1}$, $g(\pm\beta_j) = \gamma(\beta_j) \pm \beta_j/2$ and difference equation (14) associated with point z_j has the form

$$\lambda_k \gamma(\beta_k) \delta_z^2(u_k)_j - e_k \delta_z(u_k)_j = h_j^{-1} R_j, \tag{15}$$

where $\delta_z^2(v)_j = (v_{j+1} - 2v_j + v_{j-1})/h_j^2$, $\delta_z(v)_j = (v_{j+1} - v_{j-1})/(2h_j)$ denotes central difference expressions of second order and of first order for approximation of the derivatives $\partial^2 v/\partial z^2$, $\partial v/\partial z$ at the central grid point z_j, $\gamma(s) = 0.5s\,\mathrm{cth}(0.5s)$ is the Il'yn perturbation coefficient for the monotone difference scheme [4].

3.3 Let there be $z_j = z_0 = H_0 \in S_0$ and $v_0 = 1$. In this case we apply the integral form of the conservation law to the half interval $(z_0, z_{0.5})$, marked off to the right of the boundary point z_0. We get

$$\mathbf{W}_{0.5} - \mathbf{W}_0 = \int_{z_0}^{z_{0.5}} G_1 \, \mathrm{d}z, \tag{16}$$

where $\mathbf{W}_0 = \mathbf{W}|_{z=z_0}$, $\mathbf{W}_{0.5} = \mathbf{W}|_{z=z_{0.5}}$. Due to the Neumann type boundary condition (6) at $z = H_0$, the flux $\mathbf{W}_0$ is known, i.e.

$$\mathbf{W}_0 = \alpha_0\left(u_1(z_0) - \Phi_0\right).$$

As before we integrate Eq. (8) from $z_{0.5}$ to $z \in (z_0, z_1)$ and from z_0 to z_1, and we can easily derive the following two-point difference equation associated with grid point $z_0 = H_0$:

$$A_1^+\left(u_1(z_1) - u_1(z_0)\right) - \alpha_0\left(u_1(z_0) - \Phi_0\right) = R_0, \tag{17}$$

where $u_1(z_0), u_1(z_1)$ represents the value of function u_1 at z_0, z_1,

$$R_0 = \int_{z_0}^{z_1} \left(1 - A_1^+ \int_{z_0}^{z} \left(\lambda_1^*\right)^{-1} \mathrm{d}\xi\right) G_1 \, \mathrm{d}z,$$

$$A_1^+ = \lambda_1 (h_1)^{-1} g(\beta_1), \quad \beta_1 = \lambda_1^{-1} e_1 h_1, \quad \lambda_1^* = J_1 \lambda_1.$$

3.4 If $z_j = H_N \in S_N$, $v_1 = 1$, then similarly to the above we integrate Eq. (8) from $z_{j-0.5}$ to z_j. We get

$$\mathbf{W}_N - \mathbf{W}_{j-0.5} = \int_{z_{j-0.5}}^{z_j} G_N \, dz. \quad (18)$$

As in Eq. (7) we have $\nu_1 = 1$, the flux $\mathbf{W}_N = -\alpha_1(u_N(H_N) - \Phi_1)$, where $u_N(H_N)$ represents the value of the function u_N at $z_j = H_N$. We now proceed to determine the flux terms $\mathbf{W}_{j-0.5}$ in Eq. (18) using Eq. (8), by integration with respect to z from $z_{j-0.5}$ to $z \in (z_{j-1}, H_N)$ and from z_{j-1} to H_N. We obtain a two-point difference equation associated with grid point $z_j = H_N$ in the following form,

$$-\alpha_N(u_N(H_N) - \Phi_1) - A_N^-\big(u_N(H_N) - u_N(z_{j-1})\big) = R_N, \quad (19)$$

where $u_N(z_{j-1})$ represents the value of the function u_N at z_{j-1},

$$R_N = \int_{z_{j-1}}^{H_N} \left(1 - A_N^- \int_z^{H_N} \left(\lambda_N^*\right)^{-1} d\xi\right) G_N \, dz,$$

$$A_N^- = \lambda_N (h_N)^{-1} g\big(-\beta_N\big), \qquad \beta_N = \lambda_N^{-1} e_N h_N,$$

$$\lambda_N^* = J_N \lambda_N.$$

We see that difference equations (14), (15), (17) and (19) are exact approximations for solving steady-state one-dimensional boundary-value problem (3), (5)–(7) depending only on z, $(L_k(u_k) = 0, l_x = l_y = \infty)$.

4. One-dimensional exact difference scheme

Suppose that $L_k(u_k) = 0$, $u_k = u_k(z)$, $F_k = F_k(z)$, $\lambda_k; \Phi_0; \Phi_1$ are constants and the grid points are $z_k = H_k$, $k = 0, \ldots, N$. If $v_j = u_j(z_j)$ is the value of function u_j at the grid point z_j, $j = 0, \ldots, N$, then evaluating integral R_j in the right side of Eqs. (14), (17), and (19) one obtains exact one-dimensional steady-state difference scheme

$$\begin{aligned} &\nu_0 A_{j+1}^+(v_{j+1} - v_j) - \alpha_0\big(v_j - \Phi_0\big) = \nu_0 R_j, \quad j = 0 \\ &A_{j+1}^+(v_{j+1} - v_j) - A_j^-(v_j - v_{j-1}) = R_j, \quad j = 1, \ldots, N-1 \\ &\alpha_N\big(\Phi_1 - v_j\big) - \nu_1 A_j^-(v_j - v_{j-1}) = \nu_1 R_j, \quad j = N \end{aligned} \quad (20)$$

where $A_j^- = \lambda_j h_j^{-1} g(-\beta_j)$, $A_{j+1}^+ = \lambda_{j+1} h_{j+1}^{-1} g(\beta_{j+1})$, $\beta_j = (\lambda_j)^{-1} e_j h_j$, $j = 0, \ldots, N$. For solving the boundary-value problem with Dirichlet boundary condition (6) $(\nu_0 = 0)$ we have from the first difference equation of (20) that $v_0 = \Phi_0$. Analogously, we can obtain from the last difference equation of (20) that $v_N = \Phi_1$ in the case $\nu_1 = 0$.

Difference scheme (20) for $\nu_0 = \nu_1 = 1$ can be rewritten as

$$A_{j+1}^+(v_{j+1} - v_j) - A_j^-(v_j - v_{j-1}) = R_j, \quad j = 0, \ldots, N, \quad (21)$$

where,

$$A_0^- = \alpha_0 \geq 0, \quad A_{N+1}^+ = \alpha_N \geq 0, \quad v_{-1} = \Phi_0, \quad v_{N+1} = \Phi_1,$$

$$A_{j+1}^+ > 0, \quad A_j^- > 0, j = 1, \ldots, N.$$

Therefore, difference scheme (21) is monotone and has a unique solution [4]. We can consider in addition new grid points for approximation of functions u_k in layers D_k. In the case of uniform grid, we use difference equation (15). Finite-difference scheme (21) can be solved by the factorisation method for the tri-diagonal matrix (Thomas algorithm [3]).

5. Solution of one-dimensional problem

We can obtain a symmetric form of the matrix for difference schemes (20) and (21) by multiplying the jth equation with factor $\Gamma_j = \exp(-\Pi_{i=1}^j \beta_i)$, $(\Gamma_0 = 1)$. Then, from $\Gamma_{j+1} A_{j+1}^- = \Gamma_j A_{j+1}^+$ the finite-difference scheme follows

$$A_{j+1}(v_{j+1} - v_j) - A_j(v_j - v_{j-1}) = \tilde{R}_j, \quad j = 0, \ldots, N, \quad (22)$$

where,

$$A_{j+1} = \Gamma_j A_{j+1}^+, \quad j = 0, \ldots, N-1, \quad A_0 = A_0^- = \alpha_0,$$

$$A_{N+1} = \tilde{\alpha}_N = \Gamma_N \alpha_N, \quad \tilde{R}_j = \Gamma_j R_j.$$

We can solve the difference scheme (22) also in a more simple form. For this purpose from the first equation of (22) we conclude that

$$A_1(v_1 - v_0) - \alpha_1^+(v_1 - \Phi_0) = \frac{\alpha_1^+}{\alpha_0} \tilde{R}_0,$$

where $(\alpha_1^+)^{-1} = (\alpha_0)^{-1} + (A_1)^{-1}$ is the inverse value of the interaction coefficient of two layers. Furthermore, from the second equation of (22) it follows:

$$A_2(v_2 - v_1) - A_1(v_1 - v_0) = \tilde{R}_1.$$

Therefore,

$$A_2(v_2 - v_1) - \alpha_1^+(v_1 - \Phi_0) = \alpha_1^+ R_1^+,$$

where $R_1^+ = \tilde{R}_1/\alpha_1^+ + \tilde{R}_0/\alpha_0$.

Hence,

$$A_{m+1}(v_{m+1} - v_m) - \alpha_m^+(v_m - \Phi_0) = \alpha_m^+ R_m^+, \quad (23)$$

where

$$\left(\alpha_m^+\right)^{-1} = \left(\alpha_{m-1}^+\right)^{-1} + A_m^{-1} = (\alpha_0)^{-1} + A_1^{-1} + \cdots + A_m^{-1},$$

$$R_m^+ = \tilde{R}_m/\alpha_m^+ + R_{m-1}^+/\alpha_{m-1}^+$$
$$= \tilde{R}_0/\alpha_0 + \tilde{R}_1/\alpha_1^+ + \cdots + \tilde{R}_m/\alpha_m^+$$
$$m = 1,\ldots,N-1.$$

From the last equation of (22) we obtain

$$\alpha_{N-1}^-(\Phi_1 - v_{N-1}) - A_N(v_N - v_{N-1}) = \frac{\alpha_{N-1}^-}{\tilde{\alpha}_N}\tilde{R}_N,$$

where $(\alpha_{N-1}^-)^{-1} = (\tilde{\alpha}_N)^{-1} + A_N^{-1}$ is the inverse value of the interaction coefficient of two layers in opposite directions.

From Eq. (23) for $m = N-1$ it follows

$$A_N(v_N - v_{N-1}) - \alpha_{N-1}^+(v_{N-1} - \Phi_0) = \alpha_{N-1}^+ R_{N-1}^+.$$

Hence,

$$\alpha_{N-1}^-(\Phi_1 - v_{N-1}) - \alpha_{N-1}^+(v_{N-1} - \Phi_0) = R_{N-1}^\pm$$

and,

$$v_{N-1} = \frac{\alpha_{N-1}^-\Phi_1 + \alpha_{N-1}^+\Phi_0 - R_{N-1}^\pm}{\alpha_{N-1}^- + \alpha_{N-1}^+}, \tag{24}$$

where $R_{N-1}^\pm = R_{N-1}^+\alpha_{N-1}^+ + \alpha_{N-1}^-\tilde{R}_N/\tilde{\alpha}_N$.

Similarly, it can be obtained that

$$v_N = \frac{\tilde{\alpha}_N\Phi_1 + \alpha_N^+\Phi_0 - R_N^\pm}{\tilde{\alpha}_N + \alpha_N^+},$$

where $R_N^\pm = \tilde{R}_N + \alpha_N^+ R_{N-1}^+$.

For the determination of flux function $\mathbf{W}_N$, the last equation of (22) and (24) yield

$$\mathbf{W}_N = -\alpha_N(v_N - \Phi_1) = \alpha\big(\Phi_1 - \Phi_0 + R_N^+\big), \tag{25}$$

where $(\alpha)^{-1} = (\alpha_0)^{-1} + (A_1)^{-1} + \cdots + (A_N)^{-1} + (\tilde{\alpha}_N)^{-1}$ is the inverse value of the common interaction coefficient of the layers.

For Dirichlet boundary condition ($v_0 = 0$ or $v_1 = 0$) we can take $\alpha_0 = \infty$ or $\tilde{\alpha}_N = \infty$.

We can also consider the opposite direction. At $j = N$ and $j = N-1$ from Eq. (22) it follows:

$$\alpha_{N-1}^-(\Phi_1 - v_{N-1}) - A_{N-1}(v_{N-1} - v_{N-2}) = R_{N-1}^-\alpha_{N-1}^-,$$

where $R_{N-1}^- = \tilde{R}_{N-1}/\alpha_{N-1}^- + \tilde{R}_N/\tilde{\alpha}_N$.

Therefore,

$$\alpha_{N-n}^-(\Phi_1 - v_{N-n}) - A_{N-n}(v_{N-n} - v_{N-n-1}) = R_{N-n}^-\alpha_{N-n}^-, \tag{26}$$

where $(\alpha_{N-n}^-)^{-1} = (\tilde{\alpha}_N)^{-1} + (A_N)^{-1} + \cdots + (A_{N-n+1})^{-1}$, $R_{N-n}^- = \tilde{R}_{N-n}/\alpha_{N-n}^- + \tilde{R}_{N-n+1}/\alpha_{N-n+1}^- + \cdots + \tilde{R}_N/\tilde{\alpha}_N$.

From the first equation of (22) and from Eq. (26) at $n = N-1$ it follows that

$$v_1 = \frac{\alpha_1^-\Phi_1 + \alpha_1^+\Phi_0 - R_1^\pm}{\alpha_1^- + \alpha_1^+}, \tag{27}$$

where $R_1^\pm = R_1^-\alpha_1^- + \alpha_1^+\tilde{R}_0/\alpha_0$.

The value of v_0 can be obtained in the form

$$v_0 = \frac{\alpha_0^-\Phi_1 + \alpha_0\Phi_0 - R_0^\pm}{\alpha_0^- + \alpha_0},$$

where $R_0^\pm = \tilde{R}_0 + R_1^-\alpha_0^-$.

For flux value $\mathbf{W}_0$ from Eq. (27) and first equation of (22) we get the following expression

$$-W_0 = \alpha\big(\Phi_0 - \Phi_1 + R_0^-\big).$$

From expressions (23) and (26) at $m = k-1$ and $n = N-k$ it follows

$$v_k = \frac{\alpha_k^-\Phi_1 + \alpha_k^+\Phi_0 - R_k^\pm}{\alpha_k^- + \alpha_k^+}, \tag{28}$$

where $R_k^\pm = R_k^-\alpha_k^- + \alpha_k^+ R_{k-1}^+$, $k = 1,\ldots,N-1$.

6. Discrete approximation of first and second order

If $L_k(u_k) \neq 0$ and functions λ, F_k, Φ_0, Φ_1 depend on other variables, then difference scheme (20) is not exact (this is the case of 2D or 3D problems with $l_x \neq \infty$, $l_y \neq \infty$). In such cases we can obtain the accuracy of order $O(h_x + h_y + h_z)$ or $O(h_x^2 + h_y^2 + h_z^2)$, where h_x, h_y, h_z are the steps of an uniform grid in the corresponding directions. We consider different approximations for right-side function R_j in Eqs. (14), (15), (17) and (19).

6.1 To approximate R_j from Eq. (13) on a nonuniform grid we consider the following Taylor series expansions of function $P_k = -(F_k + L_k(u_k))$:

$$P_k(z) = P_k(z_j) + (z - z_j)P'_k(z_j) + O(z - z_j)^2,$$
$$z \in (z_{j-1}, z_j)$$

$$P_{k+1}(z) = P_{k+1}(z_j) + (z - z_j)P'_{k+1}(z_j) + O(z - z_j)^2,$$
$$z \in (z_j, z_{j+1})$$

where $P'_k = \partial P_k/\partial z$, $z_j = H_k$.

Then

$$R_j = h_{j+1}r\big(\beta_{j+1}\big)P_{k+1}(z_j) + h_j r\big(-\beta_j\big)P_k(z_j) + \tilde{\delta}\big(\beta_{j+1}\big)P'_{k+1}(z_j)h_{j+1}^2 - \tilde{\delta}\big(-\beta_j\big)P'_k(z_j)h_j^2 + O(h^3), \tag{29}$$

where

$h = \max(h_j, h_{j+1})$,

$$r(s) = s^{-1}(1 - g(s)) = 0.5 - s/12 + O(s^2),$$

$$\tilde{\delta}(s) = s^{-2}(1 - (1 + 0.5s)g(s)) = 1/6 - s/24 + O(s^2).$$

In the 1D case $L_k u_k = 0$, $P_k = -F_k$, $P_{k+1} = -F_{k+1}$ this expression is divided by average step $\bar{h} = (h_j + h_{j+1})/2$. We see that Eq. (29) approximates R_j to the second order in h. For the second order accuracy, also from Eq. (29), it follows

$$R_j = P_{k+1}(z_j)h_{j+1}\left(0.5 - \frac{\beta_{j+1}}{12}\right) + P_k(z_j)h_j\left(0.5 + \frac{\beta_j}{12}\right) + \frac{\left(P'_{k+1}(z_j)h_{j+1}^2 - P'_k(z_j)h_j^2\right)}{6} + O(h^3) \quad (30)$$

For the first order accuracy we get

$$R_j = \frac{(P_{k+1}(z_j)h_{j+1} + P_k(z_j)h_j)}{2} + O(h^2). \quad (31)$$

Since

$$P'_{k+1} = \frac{P_{k+1}(z_{j+1}) - P_{k+1}(z_j)}{h_{j+1}} + O(h_{j+1}),$$

$$P'_k = \frac{P_k(z_j) - P_k(z_{j-1})}{h_j} + O(h_j),$$

expressions (29) and (30) can be obtained in the form

$$R_j = h_{j+1}\left[P_{k+1}(z_{j+1}) + 2P_{k+1}(z_j)(1 - \beta_{j+1}/4)\right]/6 + h_j\left[P_k(z_{j-1}) + 2P_k(z_j)(1 + \beta_j/4)\right]/6 + O(h^3). \quad (32)$$

We see that expressions (29), (30) and (32) approximate R_j to the second order in h_j.

Evaluating R_0 from Eq. (17) we see, using a Taylor series expansion, that

$$P_1(z) = P_1(z_0) + (z - z_0)P'_1(z_0) + O(z - z_0)^2,$$

$$z \in (z_0, z_1),$$

so

$$R_0 = P_1(z_0)h_1 r(\beta_1) + P'_1(z_0)h_1^2\tilde{\delta}(\beta_1) + O(h_1^3), \quad (33)$$

or

$$R_0 = \frac{h_1}{6}\left(P_1(z_1) + 2P_1(z_0)(1 - \beta_1/4)\right) + O(h_1^3). \quad (34)$$

We see that expressions (33) and (34) approximate R_0 to the second order in h_1.

Similarly, from Eq. (19), evaluating R_N we can show that

$$P_N(z) = P_N(H_N) + (z - H_N)P'_N(H_N) + O(z - H_N)^2,$$

$$z \in (z_{j-1}, H_N),$$

so

$$R_N = P_N(H_N)h_N r(-\beta_N) - P'_N(H_N)h_N^2\tilde{\delta}(-\beta_N) + O(h_N^3), \quad (35)$$

or

$$R_N = \frac{h_N}{6}\left(P_N(z_{j-1}) + 2P_N(H_N)\right)(1 + \beta_N/4) + O(h_N^3). \quad (36)$$

We see that expressions (35) and (36) approximate R_N to the second order in h_N.

The second order of accuracy in x-, y-directions can be obtained by the central difference approximation for the derivatives in expressions (32)–(36). If $a_k \neq 0$, $b_k \neq 0$ then the monotone difference schemes can be consider [4,5].

7. Some examples

In the following examples we will discuss the applications of the finite-difference scheme (20) in 1D and in 2D cases.

Example 1

We assume that the boundary-value problem of the mathematical physics (3)–(7) for a two-layer system ($N = 2$) is a steady-state one ($e_k = 0$), with the boundary conditions at the side $x = \pm l_k$, $y = \pm l_y$:

$$\partial u_k/\partial x = \partial u_k/\partial y = 0.$$

Let there be

$$H_0 = 0, \quad H_1 = \epsilon > 0, \quad H_2 = 1, \quad v_0 = v_1 = 0,$$

$$\alpha_0 = \alpha_2 = 1, \quad F_1 = -\epsilon^{-1}, \quad F_2 = 0, \quad \Phi_0 = \Phi_1 = 0.$$

Then difference scheme (20) with three grid points $z_0 = 0, z_1 = \epsilon, z_2 = 1$ has the solutions $v_0 = v_2 = 0$, $v_1 = 0.5(\epsilon - 1)\epsilon/(\epsilon\lambda_2 + (1 - \epsilon)\lambda_1)$, where the exact solution of differential problem is at the point $z = \epsilon$.

Example 2

We assume, in addition, that through the lower surface S_0 the flux of $u_1(v_0 = 1)$ is given. Then from Eq. (20)

there follows the system of two equations:

$$\frac{\lambda_1}{\epsilon}(v_1 - v_0) - \alpha_0 v_0 = R_0,$$

$$-\frac{\lambda_2}{1-\epsilon}v_1 - \frac{\lambda_1}{\epsilon}(v_1 - v_0) = R_1,$$

where

$$R_0 = \epsilon^{-1}\int_0^{\epsilon}(1 - z/\epsilon)\,\mathrm{d}z = 0.5,$$

$$R_1 = \epsilon^{-1}\int_0^{\epsilon} z/\epsilon\,\mathrm{d}z = 0.5.$$

We obtain the exact values of the solution at the two grid points $z_0 = 0$, $z_1 = \epsilon$ in the form:

$$v_0 = -\big((1-\epsilon)\lambda_1 + 0.5\epsilon\lambda_2\big)/p,$$

$$v_1 = 0.5(\epsilon - 1)(\alpha_0\epsilon + 2\lambda_1)/p,$$

where $p = \alpha_0\lambda_1(1-\epsilon) + \lambda_2(\lambda_1 + \alpha_0\epsilon)$.

Example 3

In addition we assume that $e_1 \neq 0, e_2 = 0$. Then from Eq. (20) there follows the system of two equations:

$$\frac{\lambda_1}{\epsilon}g(\beta_1)(v_1 - v_0) - \alpha_0 v_0 = R_0,$$

$$-\frac{\lambda_2}{1-\epsilon}v_1 - \frac{\lambda_1}{\epsilon}g(-\beta_1)(v_1 - v_0) = R_1,$$

where $R_0 = r(\beta_1)$, $R_1 = r(-\beta_1)$, $\beta_1 = e_1\lambda_1^{-1}\epsilon$. We obtain exact values of the solution in the form

$$v_0 = -\big((1-\epsilon)\lambda_1\tilde{p} + \epsilon\lambda_2 R_0\big)/p,$$

$$v_1 = (\epsilon - 1)\big(\alpha_0\epsilon R_1 + \lambda_1\tilde{p}\big)/p,$$

where $p = \alpha_0\lambda_1(1-\epsilon)g(-\beta_1) + \lambda_2(\lambda_1 g(\beta_1) + \alpha_0\epsilon)$, $\tilde{p} = R_1 g(\beta_1) + R_0 g(-\beta_1)$.

If $e_1 = 0$ then we obtain the results of Example 2.

Example 4

We consider a 2D steady-state process with conditions

$$\partial u_k/\partial x|_{x=\pm l_x} = 0, \quad u_k = u_k(y, z), \quad b_k = 0,$$

$$F_k = F_k(y, z), \quad \lambda_k = \text{const}$$

and with uniform grid in the y-direction with points $y_i = -l_y + ih_y$, $i = 0, \ldots, 2N_y$ $(h_y N_y = l_y)$. Then from Eqs. (30)–(36) the finite-difference scheme follows

$$v_0\lambda_1 g(\beta_1)(v_{i,1} - v_{i,0})/h_1^2 - \alpha_0\big(v_{i,0} - (\Phi_0)_i\big)/h_1 + v_0\Big(\lambda_1 r(\beta_1) + h_1\alpha_0\tilde{\delta}(\beta_1)\Big)\delta_y^2(v_0)_i = \big(F_0^*\big)_i$$

$$i = 1, \ldots, 2N_y - 1;$$

$$\Lambda_z v_{i,j} + \Lambda_y v_{i,j} = -\big(F_j^*\big)_i,$$

$$j = 1, \ldots, N-1,\ i = 1, \ldots, 2N_y - 1;$$

$$\alpha_N\big((\Phi_1)_i - v_{i,N}\big)/h_N - v_1\lambda_N g(-\beta_N)(v_{i,N} - v_{i,N-1})/h_N^2 + v_1\Big(\lambda_N r(-\beta_N) + h_N\alpha_1\tilde{\delta}(-\beta_N)\Big)\delta_y^2(v_N)_i = \big(F_N^*\big)_i,$$

$$i = 1, \ldots, 2N_y - 1, \tag{37}$$

where

$$v_{i,j} = u_j(y_i, z_j), \qquad (p)_i = p|_{y=y_i},$$

$$\delta_y^2(p)_i = \big((p)_{i+1} - 2(p)_i + (p)_{i-1}\big)/h_y^2, \quad p = F_j;\ v_j;\ \Phi_0;\ \Phi_1,$$

$$\big(F_0^*\big)_i = v_0\Big(\tilde{\delta}(\beta_1)\alpha_0 h_1\delta_y^2(\Phi_0)_i - (F_1)_i r(\beta_1) - h_1\big(F_1'\big)_i\tilde{\delta}(\beta_1)\Big),$$

$$\big(F_N^*\big)_i = v_1\Big(\tilde{\delta}(-\beta_N)\alpha_N h_N\delta_y^2(\Phi_1)_i - (F_N)_i r(-\beta_N) + h_N\big(F_N'\big)_i\tilde{\delta}(-\beta_N)\Big),$$

$$\big(F_j^*\big)_i = \frac{(F_j)_i h_j + (F_{j+1})_i h_{j+1}}{2\overset{z_j}{h}_j},$$

$$\Lambda_y v_{i,j} = \frac{\lambda_j h_j + \lambda_{j+1}h_{j+1}}{2\overset{z_j}{h}_j}\delta_y^2(v_j)_i,$$

$$\Lambda_z v_{i,j} = \Big(\overset{z_j}{h}_j\Big)^{-1}\left(\frac{\lambda_{j+1}}{h_{j+1}}g(\beta_{j+1})(v_{i,j+1} - v_{i,j}) - \frac{\lambda_j}{h_j}g(-\beta_j) \times (v_{i,j} - v_{i,j-1})\right),$$

$$\overset{z_j}{h}_j = 0.5\big(h_j + h_{j+1}\big), \quad F_j' = \partial F_j/\partial z|_{z=z_j}.$$

The five-point difference equations (37) for $j = 1, \ldots, N-1$ have only the first order of accuracy in h_j. For the second order accuracy we need to apply nine-point difference equations with

$$\Lambda_y v_{i,j} = \overset{z_j l-1}{h_j}\Big(\lambda_{j+1}h_{j+1}\Big(\tilde{\delta}(\beta_{j+1})\delta_y^2(v_{j+1})_i + \Big(r(\beta_{j+1}) - \tilde{\delta}(\beta_{j+1})\Big)\delta_y^2(v_j)_i\Big) + \lambda_j h_j\Big(\tilde{\delta}(-\beta_j)\delta_y^2(v_{j-1})_i + \Big(r(-\beta_j) - \tilde{\delta}(-\beta_j)\Big)\delta_y^2(v_j)_i\Big)\Big),$$

$$(F_j^*)_i = \overset{z_j l-1}{h_j}\Big(h_{j+1}\Big(\tilde{\delta}(\beta_{j+1})(F_{j+1})_i + \Big(r(\beta_{j+1}) - \tilde{\delta}(\beta_{j+1})\Big)\Big(F_j^+\Big)_i\Big) + h_j\Big(\tilde{\delta}(-\beta_j)(F_{j-1})_i + \Big(r(-\beta_j) - \tilde{\delta}(-\beta_j)\Big)(F_j^-)_i\Big)\Big),$$

where $F_j^{\pm} = F_j|_{z=z_{j\pm 0}}$.

If $e_k = 0$ *then* $g = 1$, $\tilde{\delta} = 1/6$, $r = 0.5$.

If $b_k \neq 0$ *then* for monotone finite-difference scheme (37) the expression $\lambda_j\delta_y^2(p)$ is in the form

$$\gamma_y\lambda_j\delta_y^2(p) - b_j\delta_y(p),$$

where $\gamma_y = 0.5b_jh_y/\lambda_j \text{cth}(0.5b_jh_y/\lambda_j)$.

Example 5

We assume, in addition, that $u_k = u_k(x, y, z)$, $a_k \neq 0$, $b_k \neq 0$. Then for the monotone finite-difference scheme (37) the expression $\lambda_j\delta_y^2(p)$ is in the form

$$\gamma_x\lambda_j\delta_x^2(p) - a_j\delta_x(p) + \gamma_y\lambda_j\delta_y^2(p) - b_j\delta_y(p),$$

where $\gamma_x = 0.5a_jh_x/\lambda_j \text{cth}(0.5a_jh_x/\lambda_j)$.

Example 6

For solving the time-depending problem with $u_k = u_k(z, t)$, $\lambda_k = \lambda_k(t)$, $F_k = F_k(z, t)$, $d_k = d_k(t)$, $a_k = b_k = 0$, $v_0 = v_1 = 1$ the difference equations can be written in the form of (21), where

$$R_0 = -h_1\Big(F_1 r(\beta_1) + h_1 F_1'\tilde{\delta}(\beta_1)\Big) + h_1 d_1\Big(\Big(r(\beta_1) + \alpha_0 h_1\tilde{\delta}(\beta_1)/\lambda_1\Big)\dot{v}_1 - \alpha_0 h_1\dot{\Phi}_0\tilde{\delta}(\beta_1)/\lambda_1\Big),$$

$$R_N = -h_N\Big(F_N r(-\beta_N) - h_N F_N'\tilde{\delta}(-\beta_N)\Big) + h_N d_N\Big(\Big(r(-\beta_N) + \alpha_N h_N\tilde{\delta}(-\beta_N)/\lambda_N\Big)\dot{v}_N - \alpha_N h_N\dot{\Phi}_1\tilde{\delta}(-\beta_N)/\lambda_N\Big),$$

$$R_j = -\big(F_jh_j + F_{j+1}h_{j+1}\big)/2 + 0.5\big(d_jh_j + d_{j+1}h_{j+1}\big)\dot{v}_j, \quad (38)$$

$$\dot{v}_j = \partial v_j/\partial t|_{z=z_j}, \quad \dot{\Phi}_m = \partial\Phi_m/\partial t, \quad m = 0; 1.$$

The five-point difference equations for $j = 1, \ldots, N-1$ have only the first order of accuracy in h_j. For the second order accuracy we need to apply nine-point difference equations with

$$R_j = -F_j^{*}\overset{z_j l}{h_j} + d_{j+1}h_{j+1}\Big(\tilde{\delta}(\beta_{j+1})(\dot{v}_{j+1}) + \Big(r(\beta_{j+1}) - \tilde{\delta}(\beta_{j+1})\Big)(\dot{v}_j)\Big) + d_jh_j\Big(\tilde{\delta}(-\beta_j)(\dot{v}_{j-1}) + \Big(r(-\beta_j) - \tilde{\delta}(-\beta_j)\Big)(\dot{v}_j)\Big). \quad (39)$$

In the case of analytical integration, we can determine function F_j in the right side from Eqs. (13), (17) and (19). If the grid is uniform, then difference expressions (37)–(39) may be simplifed. We can obtain the system of differential equations (20), (38), (39), where can be solved with initial conditions at $t = 0$ (the so-called method of lines).

References

[1] A. Buikis, The analysis of schemes for the modelling same processes of filtration in the underground, Riga, Acta Universitatis Latviensis 592 (1994) 25–32 (in Latvian).

[2] A. Buikis, N. Ulanova, The temperature field of filtering layered stratum. Riga, Latvian Journal of Physics and Technical Sciences N5 (1995) 59–64.

[3] J. W. Thomas, Numerical partial differential equations. Finite difference methods (1995) Springer-Verlag, New York.

[4] H. Kalis, Ausarbeitung und Anwendung der spezialen numerischen Methoden zur Losung der Probleme der mathemtischen Physik, Hydrodinamik und Magnetohydrodynamik, Riga, Acta Universitatis Latviensis 588 (1993) 175–206.

[5] H. Kalis, Special finite-difference approximations for numerical solution of some linear and non linear differential equations, in: Proc. of the 2nd Conference, Numerical Modelling in Continuum Mechanics, 22–25 August, 1994, Prague, 1995, pp. 133–138.